(1) **State the Substitution theorem (for integration) and use it to evaluate the integral $\int_1^4 \frac{cos(\sqrt{t})}{\sqrt{t}}dt$.**
(5 marks)

Statement: [1] Let $J := [\alpha, \beta]$ be an interval and $\phi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on interval I containing $\phi(J)$, then

$$\int_\alpha^\beta f(\phi(t))\phi'(t)dt = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx.$$

Let $f(x) = 2\cos\ x$ and let $\phi(t) = \sqrt{t}$. Then f and ϕ are real valued continous functions on $\mathbb{R}$, in particular on $[1, 4]$. Also ϕ has a continuous derivative on $[1, 4]$. Note that $[1, 4]$ contains $\phi([1, 4])$ and $\phi'(t) = \frac{1}{2\sqrt{t}}$. Hence by the substitution theorem stated above,

$$\begin{aligned}\int_1^4 \frac{cos(\sqrt{t})}{\sqrt{t}}dt &= \int_{\phi(1)}^{\phi(4)} 2\cos x dx \\ &= 2\int_1^2 \cos x dx \\ &= 2\sin(2) - 2\sin(1).\end{aligned}$$

`Answer:` $2(\sin(2) - \sin(1))$

(2) **Let $f : [1, 2] \to \mathbb{R}$ be the function defined by $f(x) = x$ for $x \in \mathbb{Q} \cap [1, 2]$ and $f(x) = 0$ for $x \in [1, 2] - \mathbb{Q}$. Calculate the upper and lower Riemann integrals $U(f)$ and $L(f)$ of f. If f integrable?**
(6 marks)

Recall that $U(f) = \inf U(P, f)$ and $L(f) = \sup L(P, f)$ where the infimum and supremum are taken over all the partitions of $[1, 2]$ respectively and for every partition $P = \{0 = x_0, x_1, \cdots, x_n = 1\}$

$$\begin{aligned}U(P, f) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) \text{ where } M_i = \sup_{x_{i-1}\le x\le x_i} f(x) \\ L(P, f) &= \sum_{i=1}^n m_i(x_i - x_{i-1}) \text{ where } m_i = \inf_{x_{i-1}\le x\le x_i} f(x)\end{aligned}$$

By the definition of f, for every interval $(x_{i_1}, x_i) \subset [1, 2]$, $m_i = 0$. Thus $L(f) = 0$. Also, $U(f) = \frac{3}{2}$.
However, f is Riemann integrable if and only if $U(f) = L(f)$. Hence f is not Riemann integrable.

(3) **Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $\mathbb{R}$. For $n \in \mathbb{N}$, define $f_n(x) = f(x + \frac{1}{n})$ for every $x \in \mathbb{R}$. Does the sequence of functions (f_n) converge uniformly on $\mathbb{R}$?**
(6 marks)

Given that f is uniformly continuous. Let $\epsilon > 0$ be given.Then $\forall x \in \mathbb{R}$, there exists a $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Choose a large positive natural number N such that $N > \delta^{-1}$. For all $n \geq N$,

$$\begin{aligned} |x + \frac{1}{n} - x| \leq 1/N < \delta &\implies |f(x + \frac{1}{n}) - f(x)| < \epsilon \\ \text{that is, } |x + \frac{1}{n} - x| \leq 1/N < \delta &\implies |f_n(x) - f(x)| < \epsilon \end{aligned}$$

Thus for all $n \geq N$, $|f_n(x) - x| < \epsilon$, $\forall x \in \mathbb{R}$. Hence the uniform convergence.

(4) **If the partial sums s_n of the series $\sum_{n=1}^{\infty} a_n$ are bounded, then show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, and $\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)}$.**
(6 marks)

`Following the proof Dirchlet's test` Given that the partial sums $s_n = \sum_{k=1}^{n} a_k$ are bounded, that is, there exists a positive number M such that $|s_n| \leq M$. Let $s_0 = 0$.
For every $\epsilon > 0$, choose large positive $N_0 > M\epsilon^{-1}$. Then $\frac{1}{n} < \epsilon$ for all $n \geq N_0$. Hence

$$\begin{aligned} \sum_{k=1}^{n} \frac{a_k}{k} &= \sum_{k=1}^{n} \frac{1}{k}(s_k - s_{k-1}) \\ &= \sum_{k=1}^{n} \frac{s_k}{k} - \sum_{k=0}^{n-1} \frac{s_k}{k+1} \\ &= \frac{s_n}{n} + \sum_{k=1}^{n-1} s_k \left(\frac{1}{k} - \frac{1}{k+1}\right) \quad (\text{since } s_0 = 0) \\ &= \frac{s_n}{n} + \sum_{k=1}^{n-1} \frac{s_k}{k(k+1)} \end{aligned}$$

Now since the partial sums are bounded we have $|\sum_{k=1}^{n} \frac{a_k}{k}| \leq M + M(1 - \frac{1}{n}) < 2M$, that is the partial sums of the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is a bounded sequence. Thus the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges. In other words,

$$\begin{aligned} \left|\sum_{k=1}^{n} \frac{a_k}{k} - \sum_{k=1}^{n} \frac{s_k}{k(k+1)}\right| &= \left|\frac{s_n}{n} - \frac{s_n}{n(n+1)}\right| = \left|\frac{s_n}{n+1}\right| \\ &\leq M\frac{\epsilon}{M} = \epsilon \ \forall n \geq N_0 - 1 \end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \frac{a_n}{n+1}$ converges to $\sum_{n=1}^{\infty} \frac{s_n}{n(n+1)}$.

(5) **Let $f_n(x) = Arctan(nx)$. Describe the pointwise limit function f of this sequence. Show that f_n converges to f uniformly on $(0, \infty)$.**
(6 marks)

By the properties of $Arctan$: $Arctan(\frac{1}{x}) = \frac{\pi}{2} - arctan(x)$, for positive x, we have $|Arctan(nx) - \frac{\pi}{2}| = |\frac{\pi}{2} - Arctan(nx)| = |Arctan(\frac{1}{nx})|$. Since $Arctan$ is monotonically increasing funtion, we have $|Arctan(nx) - \frac{\pi}{2}| = Arctan(\frac{1}{nx}) \to 0$. Thus $\lim_{n\to\infty} f_n(x) = \frac{\pi}{2}$. Note that the convergence is not depending on $x \in (0, \infty)$. Hence the uniform convergence on $(0, \infty)$.
Also for $x < 0$, we have $arctan\frac{1}{x} = -\frac{\pi}{2} - arctan(x)$. Hence for $x < 0$, $lim_{n\to\infty} f_n(x) = \frac{-\pi}{2}$.

(6) **If a and b are positive numbers, then prove that the series $\sum_{n=1}^{\infty}(an+b)^{-p}$ converges if $p > 1$ and diverges if $p \leq 1$.**
(6 marks)

For $p \leq 0$ if the series converged, then $lim_{n\to\infty}(an+b)^{-p} = 0$ which is a contradiction.
Let $p > 0$. Note that the partial sums of the series $\sum_{n=1}^{\infty}(an+b)^{-p}$ is monotonically increasing. Hence enough to prove that the partial sums are bounded. Let $a_n = (an+b)^p$

$$
\begin{aligned}
\text{For } n < 2^k,\ s_n &= \sum_{k=1}^{n} \frac{1}{(an+b)^p} \\
&\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\
&\leq a_1 + 2a_2 + \cdots 2^k a_{2^k}
\end{aligned}
$$

$$
\text{Now } 2^j a_j = \frac{2^{j(1-p)}}{(a + \frac{b}{2^j})^p} < \frac{2^{k(1-p)}}{a^p} \implies s_n \leq a^{-p} \sum_{j=0}^{k} 2^{-k(p-1)}
$$

Since $\sum 2^{-n(p-1)}$ converges if and only if $p > 1$, the convergence of the series $\sum_n (an+b)^{-p}$ follows for $p > 1$. And for $p \leq 1$, by comparison test, the divergence follows.

References

[1] Bartle, R. G. and Sherbert, D. R., "Introduction to Real analysis", 2007

[2] Rudin, W., "Principles of Mathematical Analysis", 1976.